Homework 1

September 22, 2025

1. Prove or disprove the following claims.
   (a) $2^{\lfloor \lg n \rfloor} = \Theta(2^{\lceil \lg n \rceil})$.
   (b) $2^{2^{\lfloor \lg \lg n \rfloor}} = \Theta(2^{2^{\lceil \lg \lg n \rceil}})$.

2. List the following functions in increasing asymptotic order. Between each adjacent functions in your list, indicate whether they are asymptotically equivalent ($f(n) \in \Theta(g(n))$, you may use the notation that $f(n) \equiv g(n)$) or if one is strictly less than the other ($f(n) \in o(g(n))$ and use the notation that $f(n) \prec g(n)$).

$5n^3 + \log n$	2^n	$3^{n/2}$	$2^{n/3}$	$\sqrt{\lg n}$
$\ln n$	$2^{\sqrt{\lg n}}$	$\min\{n^2, 1045n\}$	$\sum_{i=1}^{n} i^{77}$	$n^{\ln 4}$
$\lfloor n^2/45 \rfloor$	$\lceil n^2/45 \rceil$	$n^2/45$	$\lg \sqrt{n}$	$\lg \lg n$
$\sum_{i=1}^{n} 1/i$	$\sum_{i=1}^{n} 1/i^2$	$\sum_{i=1}^{n} (i^2 + 5i)/(6i^4 + 7)$	$\ln(n!)$	$(\lg n)^{\sqrt{\lg n}}$

3. Solving recurrences. Find the asymptotic order of the following recurrence, represented in big-Θ notation.
   (a) $A(n) = 4A(\lfloor n/2 \rfloor + 5) + n^2$
   (b) $B(n) = B(n-4) + 1/n + 5/(n^2+6) + 7n^2/(3n^3+8)$
   (c) $C(n) = n + 2\sqrt{n}C(\sqrt{n})$ Hint: take $H(n) = C(n) + n$.

4. Counting Inversions: ([KT] Chapter 5.1) An inversion in an array A is a pair of indices (i, j) such that $i < j$ and $A[i] > A[j]$. Give a divide-and-conquer algorithm to count the number of inversions in an array in $\mathcal{O}(n \log n)$ time.

5. Karatsuba's Algorithm for Integer Multiplication: ([DPV] Chapter 2.1) Multiply two n-digit integers x and y in faster than $\mathcal{O}(n^2)$ time.

6. Finding the k-th Smallest Element (Quickselect): ([CLRS] Chapter 9.2) Given an unsorted array A of n distinct elements, find the k-th smallest element in expected $\mathcal{O}(n)$ time.

7. Closest Pair of Points: ([KT] Chapter 5.4) Given n planar points, find the pair with minimum Euclidean distance.

8. Suppose you are given a function $f(A, i)$ which sorts the subarray $A[i+1, i+2, \cdots, i+\sqrt{n}]$ in place (meaning the elements are re-arranged in the subarray) for any given $0 \le i \le n - \sqrt{n}$.
   (a) Design an algorithm which only calls this function f to sort a given array $A[1..n]$. How many times do you call this function? Given the asymptotic answer in $O(\cdot)$ notation. Your algorithm is not allowed to directly compare elements in A.

(b) Prove that the algorithm you design in (a) is optimal up to a constant factor. That is, argue that no other algorithm can be asymptotically better than your algorithm in terms of the number of times to call the function f.

9. Given n half planes $\{H_1, H_2, \cdots, H_n\}$, we ask for an efficient algorithm to compute their intersection. Specifically, a half plane H_i is defined by an inequality $a_i x + b_i y \leq c_i$ for three integers a_i, b_i, c_i (at least one of a_i, b_i is not zero for H_i to be well defined).

    (a) Prove that the intersection of $\{H_1, H_2, \cdots, H_n\}$ is convex with at most n boundary edges. Here a set S is convex if $\forall x, y \in S$, the points on the line segment xy are also in S.

    (b) Develop a divide-and-conquer algorithm with running time $O(n \log n)$.

10. A Toeplitz matrix is an $n \times n$ matrix $A = (a_{ij})$ such that $a_{ij} = a_{i-1,j-1}$ for $i = 2, 3, \cdots, n$ and $j = 2, 3, \cdots, n$.

    (a) Is the sum of two Toeplitz matrices necessarily Toeplitz? What about the product?

    (b) Describe how to represent a Toeplitz matrix so that two $n \times n$ Toeplitz matrices can be added in $O(n)$ time.

    (c) Give an $O(n \log n)$ algorithm for multipling an $n \times n$ Toeplitz matrix by a vector of length n. Use your representation in the previous part.